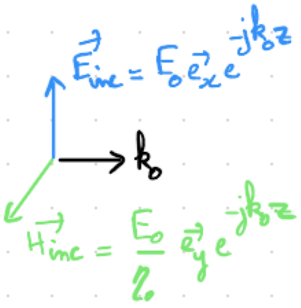
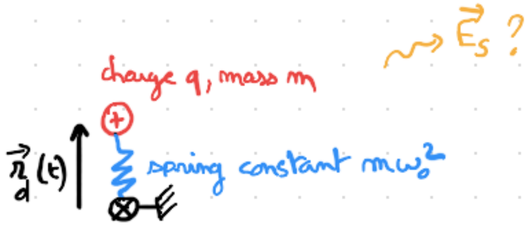
10) Dipolar scattering

1. Problem statement and basic model

Scattering problem





$$\boxed{\vec{p} = q\vec{r}_d(t)}$$ electric dipole

$$\boxed{i(t)\Delta\ell = \frac{d\vec{p}}{dt}}$$ electric current

$\Delta\ell$ = dipole length $\ll \lambda_0$

Model of an infinitesimal electric dipole

$$m\ddot{\vec{r}}_d = q\vec{E}_{loc} - m\omega_0^2\vec{r}_d - \gamma_h \vec{r}_d + \vec{F}_{rad}$$

spring

local field = field that would be there in absence of the dipole $= \vec{E}_{inc}$

absorption

radiation force = damping force the dipole exerts on itself because it radiates

2. Radiation reaction force

$$\boxed{\text{Goal: To calculate } \vec{F}_{rad}}$$

Let's forget for a moment our scattering problem and consider a small Hertzian dipole aligned along $\vec{e}_z$: $\vec{p} = \frac{I_0 \Delta l}{j\omega}\vec{e}_z$, where Δl is the length of the dipole.

1) Calculate the radiated electric field (exact)

The current density is $\vec{J}_z(\vec{r}') = I_0 \Delta l\, \delta(\vec{r}')\,\vec{e}_z \Rightarrow \vec{A} = A_z \vec{e}_z$ in Lorentz gauge.

$$\Rightarrow \vec{A_z}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3\vec{r}'\, J_z(\vec{r}')\cdot \frac{e^{-jk_0|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} = \frac{\mu_0}{4\pi} I_0\Delta l\, \frac{e^{-jk_0 r}}{4\pi r}$$

$$\Rightarrow \vec{H}(\vec{r}) = \frac{1}{\mu_0}\vec{\nabla}\times\vec{A}(\vec{r}) = \frac{I_0\Delta l}{4\pi}\vec{\nabla}\times\left(\frac{e^{-jk_0 r}}{r}\vec{e}_z\right) \qquad \vec{\nabla}\times(f\vec{a}) = \vec{\nabla}f\times\vec{a} + f\vec{\nabla}\times\vec{a}$$

$$\Rightarrow \vec{H}(\vec{r}) = \frac{I\Delta l}{4\pi}\vec{\nabla}\left(\frac{e^{-jk_0 r}}{r}\right)\times\hat{z} = \frac{I\Delta l}{4\pi}\left(-\frac{1}{r^2} - \frac{jk_0}{r}\right)e^{-jk_0 r}\,\underbrace{\vec{e}_r\times\hat{z}}_{\begin{pmatrix}1\\0\\0\end{pmatrix}\times\begin{pmatrix}\cos\theta\\-\sin\theta\\0\end{pmatrix} = \begin{pmatrix}0\\0\\-\sin\theta\end{pmatrix}}$$

$$\Rightarrow \vec{H}(\vec{r}) = \frac{I_0\Delta l}{4\pi}\left(\frac{1}{r^2} + \frac{jk_0}{r}\right)e^{-jk_0 r}\sin\theta\,\vec{e}_\varphi$$

$$\Rightarrow \vec{E}(\vec{r}) = \frac{1}{j\omega\varepsilon_0}\vec{\nabla}\times(\mu_0\vec{H}) = -j\frac{\mu_0}{\omega\varepsilon_0}\frac{I_0\Delta l}{4\pi}\left\{\frac{\vec{e}_r}{r\cancel{\sin\theta}}\frac{\partial}{\partial\theta}\left[\underbrace{\left(\frac{1}{r^2}+\frac{jk_0}{r}\right)e^{-jk_0 r}}\ \overset{\to 2\cos\theta\cancel{\sin\theta}}{\sin^2\theta}\right]\right.$$

$$\left. - \frac{\vec{e}_\theta}{r}\frac{\partial}{\partial r}\left[\underbrace{\left(\frac{1}{r}+jk_0\right)e^{-jk_0 r}}_{\left(-\frac{1}{r^2}-\frac{jk_0}{r}+k_0^2\right)e^{-jk_0 r}}\ \underline{\sin\theta}\right]\right\}$$

$$\boxed{\Rightarrow \vec{E}(\vec{r}) = -j\frac{\mu_0}{\omega\varepsilon_0}\frac{I_0\Delta l}{4\pi}\left\{\vec{e}_r\cos\theta\left(\frac{2}{r^3}+\frac{2jk_0}{r^2}\right) + \vec{e}_\theta\sin\theta\left(\frac{1}{r^3}+j\frac{k_0}{r^2}-\frac{k_0^2}{r}\right)\right\}e^{-jk_0 r}}$$

2) Take the limit $r \to 0$

To get the field that will act back on the dipole, we need to take $\theta = \frac{\pi}{2}$ and $r \to 0$, we get:

$$\vec{E}(r,\theta=\tfrac{\pi}{2}) = -\frac{j\mu_0}{\omega\varepsilon_0}\frac{I_0\Delta l}{4\pi r}k_0^2\left[1+\frac{1}{jk_0 r}-\frac{1}{k_0^2 r^2}\right]e^{-jk_0 r}\,\vec{e}_z \qquad \left[\frac{\mu_0}{\omega\varepsilon_0}k_0^2 = k_0\frac{\sqrt{\mu_0\varepsilon_0}\,\omega}{\omega\varepsilon_0} = k_0\eta_0\right]$$

$$= -j\eta_0 k_0\frac{I_0\Delta l}{4\pi r}\left[1+\frac{1}{jk_0 r}-\frac{1}{k_0^2 r^2}\right]e^{-jk_0 r}\,\vec{e}_z$$

But since $\exp[-jk_0 r] \approx 1 - jk_0 r - \frac{k_0^2 r^2}{2} + j\frac{k_0^3 r^3}{6}$, the field at $r\to 0$ is:

$$\frac{1}{r}\left[1+\frac{1}{jk_0 r}-\frac{1}{k_0^2 r^2}\right]\left[1-jk_0 r-\frac{k_0^2 r^2}{2}+j\frac{k_0^3 r^3}{6}\right] \approx \frac{1}{r}\left[\cancel{1} - jk_0 r + \cancel{\frac{1}{jk_0 r}}\ \cancel{-1}\ \overbrace{-\frac{k_0 r}{2j}}^{+\frac{jk_0 r}{2}} - \frac{1}{k_0^2 r^2} + \cancel{\frac{j}{k_0 r}} + \frac{1}{2}\right.$$

$$\left. - j\frac{k_0 r}{6}\right] = \frac{1}{r}\left[-\frac{2}{3}jk_0 r + \frac{1}{2} - \frac{1}{k_0^2 r^3}\right]$$

$$\Rightarrow \underline{\vec{E}(r\to 0)} \simeq -jk_0\vec{e}_z\frac{\eta_0 I_0\Delta l}{4\pi}\left[\boxed{-\frac{2}{3}jk_0} + \frac{1}{2r} - \frac{1}{k_0^2 r^3}\right] = \underline{(jk_0)^2\frac{\eta_0 I_0\Delta l}{6\pi}\left[1+j\frac{3}{4k_0 r}-j\frac{3}{2k_0^3 r^3}\right]\vec{e}_z}$$

3) Extract the non singular part:

$$\vec{E}_{ns} = (jk)^2 \frac{\eta_0 I_0 \Delta l}{6\pi}\,\vec{e}_z = (j\omega)^2 \mu_0 \overset{1/c_0}{\boxed{\varepsilon_0 \eta_0}} \frac{I_0 \Delta l}{6\pi}\,\vec{e}_z \Rightarrow \boxed{\vec{E}_{ns}(t) = \frac{\mu_0}{6\pi c_0}\Delta l \frac{d^2 i(t)}{dt^2}\,\vec{e}_z}$$

4) Calculate the electromotive force

$$\vec{v}_{em} = -\int \vec{E}\cdot d\vec{l} = -\frac{\mu_0}{6\pi c_0}\Delta l^2 \frac{d^2 i}{dt^2}$$

5) Calculate the radiated power

$$P_r(t) = \vec{v}_{em}(t)\, i(t) = -\frac{\mu_0}{6\pi c_0}\Delta l^2\, i \frac{d^2 i}{dt^2} \Rightarrow \boxed{P_r(t) = -\frac{\mu_0}{6\pi c_0}\frac{d\vec{p}}{dt}\cdot\frac{d^3\vec{p}}{dt^3}}$$

Radiated power

[$P_r = -\frac{\mu_0}{6\pi c_0}\ddot{\vec{p}}^2$ is wrong eg. Novotny's book]

6) Deduce the radiation reaction force

$$\int_{t_1}^{t_2} P_r(t)\,dt = -\int_{t_1}^{t_2} W(\vec{F}_r) = -\int_{t_1}^{t_2} \underbrace{\vec{F}_r(t)}\cdot\dot{\vec{r}}_d\,dt$$

$$\text{But} \int_{t_1}^{t_2} P_r(t)\,dt = -\int_{t_1}^{t_2} \frac{\mu_0}{6\pi c_0}\overset{q\vec{r}_d}{\frac{d\vec{p}}{dt}}\cdot\frac{d^3\vec{p}}{dt^3} = -\int_{t_1}^{t_2} \underbrace{\frac{\mu_0 q^2}{6\pi c_0}\frac{d^3\vec{r}_d}{dt^3}}\cdot\dot{\vec{r}}_d$$

$$\Rightarrow \boxed{\vec{F}_r(t) = \frac{\mu_0 q^2}{6\pi c_0}\dddot{\vec{r}}}$$

Abraham Lorentz radiation reaction force

3. Polarizability

$m\ddot{\vec{r}}_d = q\vec{E}_{loc} - m\omega_0^2\vec{r}_d - \gamma_L\dot{\vec{r}}_d + \frac{\mu_0 q^2}{6\pi c_0}\dddot{\vec{r}}_d$. For $\vec{E}_{loc} \sim e^{j\omega t}$ and for small damping we can assume $\dddot{\vec{r}}_d \approx -\omega_0^2\dot{\vec{r}}_d$

$$\left[-\omega^2 m + m\omega_0^2 + j\omega\gamma_L + j\omega^3\frac{\mu_0 q^2}{6\pi c_0}\right]\vec{r}_d = q\vec{E}_{loc}$$

$$\Rightarrow \boxed{\vec{p} = \alpha_e\vec{E}_{loc} \quad \text{with} \quad \alpha_e^{-1} = \frac{1}{q^2}\Big[\underbrace{m(\omega_0^2-\omega^2)}_{\text{resonant part}} + j\big(\underbrace{\omega\gamma_L}_{\text{absorption loss}} + \underbrace{\omega^3\frac{\mu_0 q^2}{6\pi c_0}}_{\text{radiation loss}}\big)\Big]}$$

electric polarizability

4. Extracted power

Poynting's theorem $\Rightarrow P_{ext} = \frac{1}{2}\mathrm{Re}\left[\int_V \vec{J}^* \cdot \vec{E}_{loc}\, dV'\right]$ with $\vec{J} = j\omega\vec{p}\,\delta(\vec{r}\,') = j\omega\alpha\vec{E}_{loc}\,\delta(\vec{r}\,')$

$$\Rightarrow P_{ext} = \frac{1}{2}\mathrm{Re}\Big[-j\omega\underbrace{\alpha^*}_{\alpha_R - j\alpha_I}|\vec{E}_{loc}|^2\Big] \Rightarrow \boxed{P_{ext} = -\frac{\omega}{2}\mathrm{Im}[\alpha]\,|\vec{E}_{loc}|^2}$$

passivity requires $\mathrm{Im}(\alpha) < 0$.

5. Scattered power

Direct far field integration. If $\vec{J}(\vec{r}\,') = j\omega p\,\delta(\vec{r}\,')\vec{z}$, the far fields are found as (use your favorite method, or check EE-201 channel on SWITCHtube, video 30): with $g(r) = \exp[-jk_0 r]/(4\pi r)$

$\vec{E}^{ff} = -\omega p k_0 \eta_0 g(r)\sin\theta\,\vec{e}_\theta$ and $\vec{H}^{ff} = -\omega p k_0 g(r)\sin\theta\,\vec{e}_\varphi$

$\Rightarrow \vec{S}^{ff} = \frac{\omega^2|p|^2}{2}\eta_0 k_0^2|g(r)|^2\sin^2\theta\,\vec{e}_r \qquad \omega^2\eta_0 k_0^2 = k_0^4\frac{\cancel{\sqrt{\mu_0}}\sqrt{\varepsilon}}{\cancel{\sqrt{\varepsilon_0}}\sqrt{\mu_0}\,\varepsilon_0} = \frac{k_0^4}{\eta_0\varepsilon_0^2}$

$$\Rightarrow P_{rad} = \frac{|p|^2}{2\eta_0}\frac{k_0^4}{16\pi^2\varepsilon_0^2}2\pi\int_0^\pi \frac{1}{\cancel{r^2}}\cancel{r^2}\sin^3\theta\, d\theta = \boxed{|p|^2\frac{k_0^4}{6\pi\eta_0\varepsilon_0^2} = P_{rad}}$$

6. Power conservation

$$P_{ext} \geqslant P_{rad} \Rightarrow -\frac{\omega}{\cancel{2}} \operatorname{Im}(\alpha) \cancel{|E_{loc}|^2} \geqslant |\alpha|^2 \cancel{|E_{loc}|^2} \frac{k_0^4}{\overset{\cancel{12}}{6}\pi \eta_0 \varepsilon_0^2} \Rightarrow -\frac{\operatorname{Im}\alpha}{|\alpha|^2} \geqslant \frac{2}{\underset{\downarrow\, k_0/\sqrt{\mu_0\varepsilon_0}}{\omega}} \frac{k_0^4}{12\pi\eta_0\varepsilon_0^2} = \frac{k_0^3}{6\pi\varepsilon_0}$$

$$-\frac{1}{2}\frac{\alpha-\alpha^*}{\alpha\alpha^*} = \frac{1/\alpha - 1/\alpha^*}{2} \Rightarrow$$

$$\boxed{\begin{aligned} &\operatorname{Im}(\alpha^{-1}) \geqslant \frac{k_0^3}{6\pi\varepsilon_0} && \text{power conservation (Sipe-Kranendonk)} \\ &\operatorname{Im}(\alpha) \leqslant 0 && \text{passivity} \end{aligned}}$$

Rq: 1 - $\dfrac{k_0^3}{6\pi\varepsilon_0} = \operatorname{Im} G(\vec{r}=\vec{r}_0)$ = LDOS in free space [Novotny's book]

2 - $\operatorname{Im}(\alpha^{-1}) = \dfrac{k_0^3}{6\pi\varepsilon_0} \Leftrightarrow P_{ext} = P_{scat} \Leftrightarrow P_{abs} = 0 \Leftrightarrow$ lossless $\Leftrightarrow$ ideal case

3 - We found : $\operatorname{Im}(\alpha^{-1}) = \dfrac{\omega}{q^2}\gamma_L + \underbrace{\omega^3 \dfrac{\mu_0}{6\pi c_0}}_{\text{term from radiation reaction}}$

$$\frac{\omega^3\mu_0}{6\pi c_0} = \frac{k_0^3 \cancel{\sqrt{\varepsilon_0\mu_0}}\,\cancel{\mu_0}}{6\pi\cancel{\sqrt{\varepsilon_0\mu_0}}\,\varepsilon_0\cancel{\mu_0}} = \frac{k_0^3}{6\pi\varepsilon_0}$$

⇒ power conservation is enforced by the radiation reaction force!

BEAUTIFUL!

6. Scattered field

Now we solve the problem! $\vec{E}_{inc} = \vec{E}_{loc} = E_0 \vec{e}_x e^{-jk_0 z}\Big|_{z=0} \Rightarrow \vec{p} = \alpha \vec{E}_{loc} \sim \vec{e}_x$

$$\boxed{\vec{p} = \frac{E_0}{\frac{m}{q^2}(\omega_0^2-\omega^2) + j\left(\frac{k_0^3}{4\pi\varepsilon_0} + \frac{\omega}{q^2}\gamma_L\right)} \vec{e}_x = p\,\vec{e}_x}$$

electric dipole induced by the incident field

$\vec{E}_s$ = field radiated by the induced $\vec{p}$ or associated $\vec{J} = j\omega\vec{p}\,\delta(\vec{r}')$

In far field we get: [EE-201 or your favorite method]

$$\boxed{\begin{aligned} \vec{E}_s^{ff} &= \omega p k_0 \eta_0 g(r) \left(\cos\theta\cos\varphi\,\vec{e}_\theta - \sin\varphi\,\vec{e}_\varphi\right) \\ \vec{H}_s^{ff} &= \omega p k_0 g(r) \left(\cos\theta\cos\varphi\,\vec{e}_\varphi + \sin\varphi\,\vec{e}_\theta\right) \end{aligned}}$$

Scattered far fields